Final Examination M. Math II Year (Differential Geometry II) 2014

Attempt all questions. Each question carries 6 + 6 = 12 marks. Books and notes maybe consulted Results proved in class, or propositions (with or without proof) from the class notes maybe used after quoting them. Results from exercises, however, must be proved in full if used.

1. Let M be the subset of $\mathbb{R}^3$ defined by:

$$M = \{(x, y, z) : x^2 + y^4 + z^6 = 1\}$$

(i): Prove that M is a smooth 2-dimensional submanifold of $\mathbb{R}^3$, and compute its tangent space at a point $x = (a, b, c)$ on it.

(ii): Let ω be the restriction of the 1-form $xz\,dx + 2y^3z\,dy$ to M. Show that ω is a closed form on M. Is it exact?

2. Consider the smooth vector field $X(x, y) = (-2x, 2y)$ on $\mathbb{R}^2$.

(i): Find the 1-parameter family of diffeomorphisms corresponding to X.

(ii): Let $\omega = 2x dx \wedge dy$. Compute the Lie derivative $L_X\omega$.

3. Consider the torus T^2 as the surface obtained by revolving the circle of unit radius centred at $(2, 0)$ in the xz-plane, about the z-axis.

(i): Compute the second fundamental form of T^2 in the coordinate chart defined by

$$h(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u)$$

for $(u, v) \in (0, 2\pi) \times (0, 2\pi)$.

(ii): Compute the mean and scalar curvatures of T^2 in the chart above, as functions of u and v.

4. (i): Let H^2 be the hyperbolic plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the hyperbolic metric $y^{-2}(dx^2 + dy^2)$. Let $X = A\partial_x + B\partial_y \in T_{(0,a)}H^2$ be a tangent vector, and $c(t) = (t, a)$ be the horizontal line starting at $(0, a)$. Compute the parallel-transported vector P_tX at (t, a).

(ii): Let σ be a smooth curve* in a Riemannian manifold M. Suppose that σ is of minimal length among all piecewise-smooth curves joining $x = \sigma(0)$ to $y = \sigma(1)$. Prove that σ is a geodesic (in the Levi-Civita connection, of course).

5. Consider the Lie group $G = SO(3) = \{A \in GL(3, \mathbb{R}) : AA^t = A^tA = I,\ \det A = 1\}$.

(i): Write down $so(3) := \mathrm{Lie}(SO(3)) = T_I(SO(3))$, and prove that the symmetric bilinear form on $so(3)$ defined by $\langle X, Y\rangle = -\mathrm{tr}\, XY$ is positive definite, and the left-invariant metric on $SO(3)$ arising from it is bi-invariant.

(ii): Consider the elements of $so(3)$ given by:

$$X = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}$$

Compute the sectional curvature $k(X, Y)$ of the plane spanned by X and Y in $so(3)$, in the bi-invariant metric defined in (a) above.

* of unit speed